Computation of Instantaneous Bit Error Probability from Log-Likelihood Ratio

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Abstract This paper presents a simple and accurate method for computing the Bit Error Probability (BEP) from Log-Likelihood Ratio (LLR). We emphasize the difference between BEP and LLR in order to rule out the direct use of LLR as a measure of transmission quality. This in turn leads to a simple algebraic expression for mapping LLR to BEP. In effect, we derive a reliability metric that gives the probability that a decoded bit is incorrect, and works as a stochastic time-series for Adaptive Modulation and Coding (AMC), Channel Quality Indication (CQI), and physical layer procedures.

Keywords Bit Error Probability (BEP), Log-Likelihood Ratio (LLR), Adaptive Modulation/Coding (AMC), Channel Quality Indication (CQI), Hybrid Automatic-Repeat-Request (HARQ).

1. Introduction

Researchers have used reliability information from soft-decision channel decoders for a variety of adaptation and control techniques. As nearly all wireless systems have some form of channel coding (often in the form of turbo, convolution, or low-density parity check codes), and that soft-decision decoders are almost always used to decode channel codes, the use of soft-output reliability information for the measurement and adjustment of radio resources is a subject of great significance in physical layer and cross-layer design. In fact, abundant literature exists on application of soft-output reliability information for the design or enhancement of Hybrid Automatic-Repeat-ReQuest (HARQ) procedures, Adaptive Modulation and Coding (AMC) algorithms, and Channel Quality Indication (CQI) methods.

In [1], for example, the use of error rate estimates for AMC in GSM was proposed. It was found that soft reliability information lead to much more effective adaptation than SNR-based methods in the presence of SNR inaccuracy. In [2], an unbiased estimator for PER and BER from log-likelihood ratio was analytically derived which removed dependency on the estimation of SNR and its uncertainty. In [3], a HARQ scheme was described based on codeword reliability metrics derived from soft-output Viterbi decoders.

More recently, routing and relay metrics schemes based entirely on reliability information were proposed, [4][5]. These approaches pave the way for cross-layer design of efficient mobile ad hoc networks with substantially lower packet delay than distance-based routing algorithms.

Most of these methods, however, focus on first mapping the Log-Likelihood Ratio (LLR) to a probability of bit error (BEP) (or estimating error rates from LLR), and then using simple time-averages of BEP for adaptation and control.

In this paper, we take a different approach. We treat the LLR and BEP as random variables, or stochastic time-series, and we focus on determining the probability distribution function (PDF) of each metric. We believe this has several advantages. First, sample averages tend to obscure important statistics of the time-series. The sample realizations of two different time-series may have the same mean, yet highly different characteristics. Second, and perhaps more importantly, under certain channel conditions, both LLR and BEP may turn out to be non-stationary and consequently not have constant mean values. The fact remains that while the mapping of LLR to BEP is well-known in the literature, to the best of our knowledge, no attempt has been made to derive the PDF of BEP by analysis. This is the principal contribution of this paper. We provide closed-form expressions for the PDF of both LLR and BEP, and we show simulation results to demonstrate the accuracy of our method. We also clarify the conditions under which LLR can be used instead of BEP for possible adaptation and control of radio resources.

2. The LLR and its Probability Density Function

The Log-Likelihood ratio is defined as;

$$\lambda_i = \log \frac{P(b_i = 1 | y_i)}{P(b_i = 0 | y_i)} \qquad (1)$$

where b_i is the ith transmitted bit and y_i is the corresponding received signal sample. For binary phase-shift keying (BPSK) signaling in additive white Gaussian noise (AWGN) $y_i = u_i + n_i$ where $u_i = \pm\sqrt{E_b}$ is the transmitted signal, and n_i is Gaussian noise of average power σ_n. We first derive the PDF of λ_i as follows. Using Bayes rule, equation (1) can be written as;

$$\begin{aligned}\lambda_i &= \log \frac{P(y_i | b_i = 1) P(b_i = 1) / P(y_i)}{P(y_i | b_i = 0) P(b_i = 0) / P(y_i)} \\ &= \log \frac{P(y_i | b_i = 1)}{P(y_i | b_i = 0)}\end{aligned} \qquad (2)$$

Here we assumed that $P(b_i = 1) = P(b_i = 0)$. Applying the assumption that the noise PDF is Gaussian in (2) we get,

$$\begin{aligned}\lambda_i &= \log \frac{\frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-(y_i - \sqrt{E_b})^2 / 2\sigma_n^2 \right]}{\frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-(y_i + \sqrt{E_b})^2 / 2\sigma_n^2 \right]} \\ &= \frac{2\sqrt{E_b}}{N_o/2} y_i \\ &= \frac{4\sqrt{E_b}}{N_o} (u_i + n_i)\end{aligned} \qquad (3)$$

Thus, the PDF of LLR $p(\lambda_i | u_i)$ conditioned on the transmitted signal u_i, can be calculated from (3) to be;

$$p(\lambda_i | u_i) = \frac{1}{\sqrt{2\pi} \cdot \left(2\sqrt{2E_b/N_o} \right)} \exp\left[-\frac{(\lambda_i - \frac{4\sqrt{E_b}}{N_o} u_i)^2}{16E_b/N_o} \right] \qquad (4)$$

Equation (4) shows that $p(\lambda_i | u_i) \sim N\left(\frac{4E_b}{N_o} \operatorname{sgn}(u_i), \frac{8E_b}{N_o} \right)$, i.e. a normally distributed random variable with mean and variance as indicated. With $\gamma = E_b/N_o$, we can rewrite (4) as follows,

$$p(\lambda_i | u_i) \sim N\left(4\gamma \operatorname{sgn}(u_i), 8\gamma \right) \qquad (5)$$

The PDF for LLR $p(\lambda_i)$ can be obtained from (5).

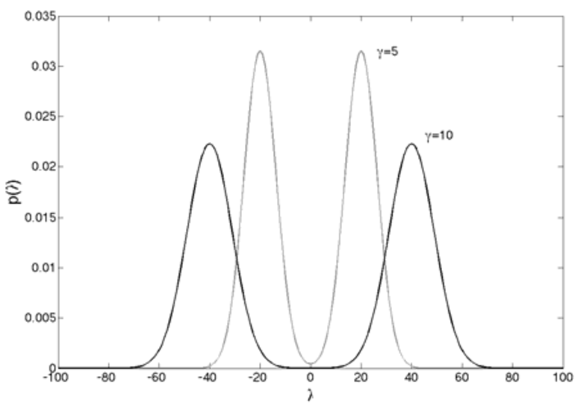

Figure 1 Plots of $p(\lambda_i)$ for $\gamma = 5$ and $\gamma = 10$.

The PDF of $p(\lambda_i)$ for two values of SNR is shown in Figure 1. To account for the conditionality upon u_i, we show two overlapping normal distributions: one for $u_i = +\sqrt{E_b}$ and another for $u_i = -\sqrt{E_b}$. The PDF of the LLR calculated from simulations over an AWGN channel with three E_b/N_o values have been shown in Fig. 2.

It is now possible to derive the expression for the probability of bit error P_b. This is calculated as follows;

$$
\begin{aligned}
P_b &= P(\lambda_i < 0 | b_i = 1) P(b_i = 1) + P(\lambda_i > 0 | b_i = 0) P(b_i = 0) \\
&= 0.5 P(\lambda_i < 0 | u_i = +\sqrt{E_b}) + 0.5 P(\lambda_i > 0 | u_i = -\sqrt{E_b}) \\
&= Q\left(\sqrt{2\gamma}\right)
\end{aligned}
\tag{6}
$$

The probability of bit error in (6) is consistent with the theoretical probability of bit error for BPSK.

3. The BEP and its PDF

Next we calculate the BEP, ε_i, for the i^{th} received bit b_i given the observed value of LLR, λ_i,

$$\varepsilon_i = \min(p_0, p_1) \tag{7}$$

where $p_0=P(b_i=0)$ and $p_1=P(b_i=1)$. Also, $p_0 = 1 - p_1$ and $p_1 = e^{\lambda_i} / \left(1 + e^{\lambda_i}\right)$. Thus,

$$\varepsilon_i = \frac{1}{1 + e^{|\lambda_i|}} \tag{8}$$

Since the PDF of λ_i is known, the PDF of BEP, $p(\varepsilon_i)$, can also be found by change of variable. By using algebraic manipulation, it can be shown that:

$$
\begin{aligned}
p(\varepsilon_i) &= \frac{1}{\sqrt{2\pi}\sigma} \cdot \frac{1}{\varepsilon_i (1-\varepsilon_i)} \Big\{ \exp\left[-\left(\ln(1/\varepsilon_i - 1) + \mu\right)^2 / 2\sigma^2 \right] \\
&\qquad + \exp\left[-\left(\ln(1/\varepsilon_i - 1) - \mu\right)^2 / 2\sigma^2 \right] \Big\} \\
&\approx \left(1/\varepsilon_i\right) \left[N\left(-\mu, \sigma^2\right) + N\left(\mu, \sigma^2\right) \right]
\end{aligned}
\tag{9}
$$

for $0 \le \varepsilon_i \le 0.5$. In (9), we approximated the PDF for $\varepsilon_i \le 0.1$ in which case the PDF, in logarithmic scale, approaches the sum of two normal distributions $N(\pm\mu,\sigma^2)$ scaled by $1/\varepsilon_i$. The parameters μ and σ are functions of the SNR γ and are specified in (5).

We verified the PDF $p(\varepsilon_i)$ in (9) numerically in MATLAB. This was done by first generating samples of λ_i according to (4) for $\gamma=3$*. These were then used to generate samples of ε_i according to (8). The histogram of ε_i was obtained and normalized to the total number of samples (10^4 in this case). This is shown in Fig. 3. Note that, the PDF has been shown only for $\varepsilon_i < 2\times10^{-9}$ in order to see the PDF around its peak value clearly.

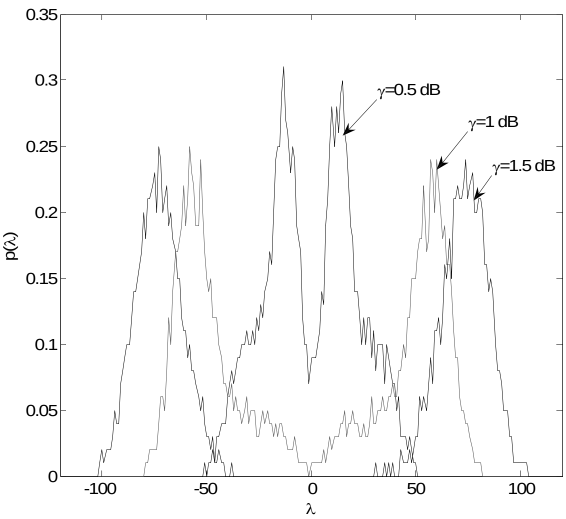

Figure 2 The measured PDF of $p(\lambda)$ from simulations in AWGN channel.

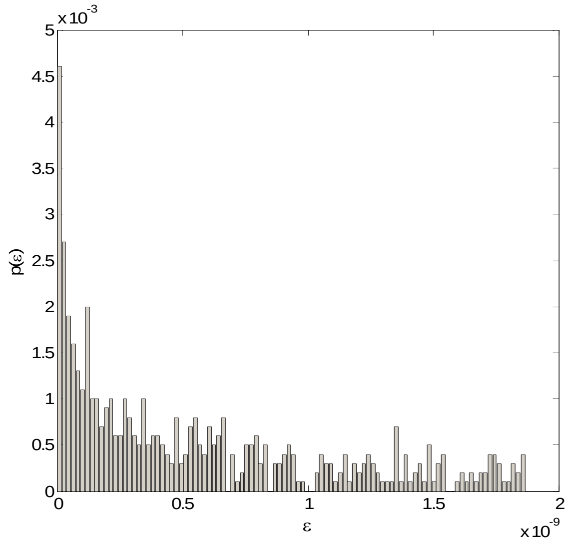

Figure 3 Normalized histogram of numerically generated ε_i.

Next we used (9) to calculate the probability of ε_i falling in each bin of the histogram in Fig. 3. This was done in MATLAB by numerical integration of (9) over the bin intervals of the histogram. The result is shown in Fig. 4.

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* Lager values would lead to extremely small numbers for $p(\varepsilon_i)$.

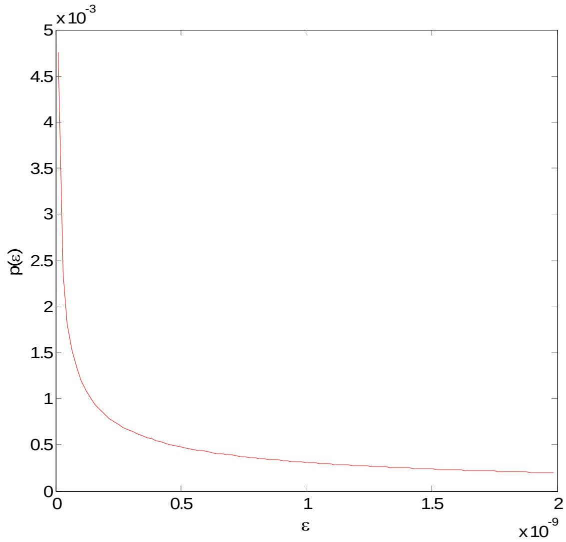

Figure 4 Calculated probabilities of ε_i for the numerically generated data. (This is the theoretical calculation of what is on Fig 3.)

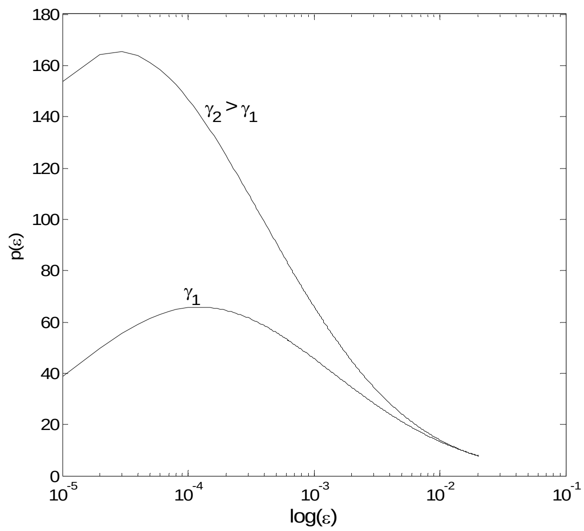

Figure 5 Two examples of $p(\varepsilon_i)$ showing the effect of increased SNR γ. For clarity logarithmic scale is used for ε_i.

4. Effect of SNR on BEP

In this Section, we first show the effect of SNR on $p(\varepsilon_i)$ based on the analytical result of (9). Then, we show that the same behavior is also confirmed by simulation.

We numerically evaluated the effect of SNR on $p(\varepsilon_i)$ for two distinct values of γ according to (9). This is shown in Fig. 5. It can be seen from Fig. 5 that the peak of $p(\varepsilon_i)$ and also the mass of the PDF move toward smaller ε_i as the SNR γ is increased. These are noticeable changes that can be quickly detected by monitoring the BEP.

The effect of γ on $p(\varepsilon_i)$ was further verified by computer simulations in a more realistic setting. In this case, we used a 16-QAM system with a standard rate-1/3 parallel concatenated convolutional turbo code. The transmitted signal was only corrupted by AWGN and subsequently detected by an iterative soft-decision decoder based on the Max-Log-MAP algorithm. The LLRs from the 8th iteration were used in (8) to compute the BEP. The results of the simulation for E_b/N_o values of 0.5 dB, 1.0 dB, and 1.5 dB are shown in Fig. 6.

The results in Fig. 6 correspond to E_b/N_o values of 0.5 dB, 1.0 dB, and 1.5 dB. The results for E_b/N_o=1.5 dB exhibited a sharp peak at $\varepsilon_i \approx 10^{-15}$ with rarely any other values observed. For E_b/N_o=1.0 dB, a sharp peak was observed at $\varepsilon_i \approx 10^{-3}$ while other values were also observed as shown in Fig. 7. Decreasing the SNR further to E_b/N_o=0.5 dB resulted in a peak at $\varepsilon_i \approx 10^{-2}$ while frequency of observing other values of ε_i increased. The measured BER for the above in the order or decreasing E_b/N_o were 0, 3.6×10^{-4}, and 1.7×10^{-3}, respectively. Additionally, the calculated averages of ε_i for the above were 1.2×10^{-15}, 1.1×10^{-3}, and 9.0×10^{-3}, respectively.

5. Approximation of BEP

For the calculation of $\varepsilon_i = \min(p_0, p_1)$ according to (8), we require to calculate exponentials. We can apply an approximation

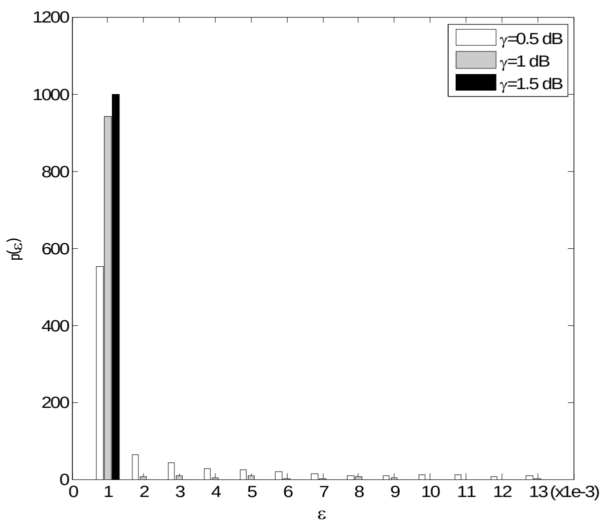

Figure 6 Examples of actual $p(\varepsilon)$ obtained from simulation

to (8) to get:

$$\varepsilon_i = e^{-|\lambda_i|} \qquad (10)$$

for $e^{|\lambda_i|} >> 1$. This is a reasonable assumption considering the PDF of λ_i as shown in Fig. 2.

Additionally, one can work the logarithm of ε_i so the quality metric becomes:

$$\log(\varepsilon_i) = -|\lambda_i| \qquad (11)$$

In (11), the LLRs can directly be applied as an adaptation metric.

6. Conclusions

In this paper we first derived the PDF of LLR for a BPSK signal, and subsequently applied it to derive the PDF of BEP. To the best of our knowledge this has not been done before. These are given by the normal distribution unction *N*(.,.) and the SNR, γ, in Table 1.

Table 1 Summary of PDFs derived in this paper.

Acronym	Symbol	Probability density function
LLR	λ_i	$p(\lambda_i) = 0.5N(-4\gamma,8\gamma) + 0.5N(4\gamma,8\gamma)$
BEP	ε_i	$p(\varepsilon_i) \approx (1/\varepsilon_i)[N(-4\gamma,8\gamma) + N(4\gamma,8\gamma)]$ **

It should be noted that the calculation of the PDF for LLR in this paper has assumed only LLR for uncoded BPSK. However, PDF for higher order modulation with coding can also be derived. The calculation of BEP from LLR values and the knowledge of the corresponding BEP PDF can be very useful for deriving CQI for adaptive modulation and coding, or for application in conjunction with HARQ. We also have shown an approximate relationship between BEP and LLR in Section 5. The approximation is useful for mapping LLR to BEP at high SNR. But it also shows that the use of LLR as a quality metric is not equivalent to BEP unless the SNR is high enough for the approximation in (10) and (11) to hold.

** For an exact PDF see (9)

7. References

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